Helicity, topology, and force-free fields

Gerald E. Marsh
Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439
(Received 8 December 1992)

Concepts from topology are increasingly finding utility in magnetohydrodynamics. This paper gives an example of how the connectivity of the domain and the gauge freedom of the vector potential can play an important role in computing the helicity of twisted magnetic fields used in several areas of astrophysics, particularly solar physics. By computing the relative helicity of a simple magnetic field configuration used to model solar prominences, it is shown that helicity can have a nonlocal character. This necessitates a reexamination of its conventional physical interpretation. The magnetic energy is also discussed.

PACS number(s): 52.30.Bt, 96.60.Se, 03.50.De

INTRODUCTION

The concept of the helicity of a magnetic field, and its topological interpretation, was first introduced in a magnetohydrodynamical context by Moffat [1]. Since then it has had a significant application in many areas of physics including the magnetic configurations used to model eruptive solar flares [2].

The helicity of a magnetic field can be expressed as

$$H=\int_V \mathbf{A}\cdot\mathbf{B}\,dV \ . \tag{1}$$

It is a constant of the motion of the equations of nondissipative magnetohydrodynamics. Helicity can be identified with the asymptotic Hopf invariant introduced by Arnold [3], and defined by him as the mean value of the asymptotic linking number of a pair of field lines. He noted that if the field lines correspond to a unitary vector field that is tangent to the Hopf bundle, the asymptotic Hopf invariant and the classical Hopf invariant coincide. The classical Hopf invariant [4] itself (see the Appendix) was first shown to be expressible as a volume integral by Whitehead [5]. The asymptotic Hopf invariant is applicable to cases where the magnetic field may have a complicated topology and the field lines may not close (although it should be noted that to define the asymptotic Hopf invariant Arnold introduces "short curves" that do indeed close the field lines, but he also shows that the asymptotic linking number is independent of this family of short curves).

Using a geometric invariant of a space curve, called the writhing number [6], Berger and Field [7] decomposed the helicity of a magnetic field into the sum of "twist" and "kink" helicities and, based on the work of Fuller [8], defined the helicity of open field structures. They made use of a theorem [9] from knot theory [10] that states that the linking number of two curves X and Y without common points can be written as the sum of the twist number T_w and the writhing number W_R,

$$L_{XY}=T_W+W_R \ . \tag{2}$$

This makes sense, for example, if one considers X and Y to be the edges of a ribbon; the example given by Berger and Field identifies X with the central axis of a flux rope and Y with a field line winding about this axis. The topological interpretation of helicity in terms of the Gauss linking number and its limiting form, the Călugăreanu invariant, has been extensively discussed by Moffatt and Ricca [11].

Moffat [12] has also noted that knotted flux tubes may always be decomposed into two or more linked tubes. Thus a flux tube twisted by 4π is equivalent to two linked but untwisted flux tubes. This can be seen as follows: Since the integrand in Eq. (1) can be written as $\mathbf{A}\cdot\mathbf{B}dSdl=\mathbf{A}\cdot d\mathbf{l}$ BdS, where $d\mathbf{l}$ is along the axis of the flux tube, the helicity introduced by the twist is

$$H_T=\Phi\int \mathbf{A}\cdot d\mathbf{l}=2\Phi^2 \ , \tag{3}$$

where Φ is the constant flux in the tube. Let the total conserved helicity be given by the sum of the twist helicity H_T, the kink helicity H_K, and the link helicity H_L. Following Berger and Field, the "twist" helicity can be shared with a "kink" helicity [13] [Fig. 1(b)]. In Fig. 1(c) the crossing is switched by introducing a link helicity, and in Fig. 1(d) the kink is eliminated by straightening

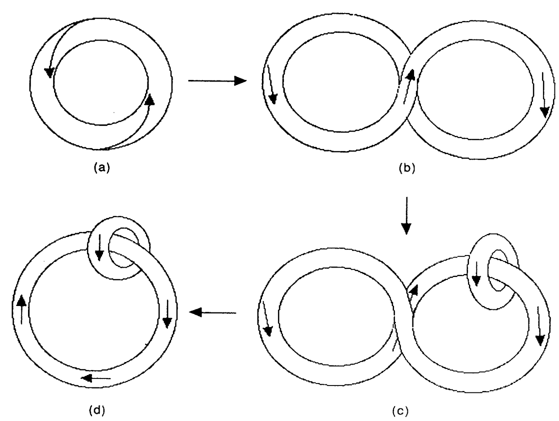

FIG. 1. The sequence of manipulations that exchange a twist helicity of $2\Phi^2$ for a link helicity. (a) $H_T=2\Phi^2$, $H_K=0$, $H_L=0$; (b) $H_T=\Phi^2$, $H_K=\Phi^2$, $H_L=0$; (c) $H_T=\Phi^2$, $H_K=-\Phi^2$, $H_L=2\Phi^2$; (d) $H_T=0$, $H_K=0$, $H_L=2\Phi^2$.

out the figure eight, leaving only a link helicity. One can convince oneself of the validity of these exchanges by playing with a piece of ribbon.

Physically, helicity has been interpreted as a measure of the degree of linkage of the field lines. The example given below is important because it illustrates that this definition cannot be taken too literally. Helicity can be nonlocalizable in the same sense that the magnetic energy of a magnetostatic field is not localizable. It will be shown that introducing a uniform twist in a previously untwisted flux tube does not increase the helicity in the flux tube itself, even though the field lines there are clearly linked. Instead, the twist creates a link helicity. Physically this results from the fact that the twist creates an axial current that is the source for a potential field that links the flux tube, thus giving rise to the link helicity.

A UNIFORMLY TWISTED FLUX TUBE

Even though they are often used for heuristic purposes, uniformly twisted flux tubes, where the $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\phi}}$ components of the field are constant *and no current is present,* cannot exist in nature. This is clear from the fact that an axial current *must* be present since $\oint \mathbf{B}\cdot d\mathbf{l}$, where the integral is about the tube, does not vanish. Thus, twisted flux loops will have currents present that will in general give rise to additional magnetic fields that link the twisted loop. An exception to this is the case where the axial current is compensated by a return surface current.

A uniformly twisted flux tube, albeit with the $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\phi}}$ components of the field decreasing monotonically with increasing radius, does exist. It is force free [14] in the sense that it satisfies

$$\nabla\times\mathbf{H}=\alpha\mathbf{H}\ , \tag{4}$$

where α is a scalar function of position. It is readily verified that the field [15,16]

$$\mathbf{B}=\left[0,\frac{B_0}{(1+r^2/a^2)}\frac{r}{a},\frac{B_0}{(1+r^2/a^2)}\right]\ , \tag{5}$$

where a is a constant, satisfies Eq. (4) with α being given by

$$\alpha=\frac{2}{a(1+r^2/a^2)}\ . \tag{6}$$

That the field is uniformly twisted can be seen from the fact that $B_\phi/B_z=r/a$. If the angle θ is defined with respect to the z axis, $\tan\theta=B_\phi/B_z=r\,d\phi/dz$. Therefore, the z distance traversed is $z=a\int d\phi$, independent of r. Note that the case of a constant axial field is given by $a\to\infty$.

HELICITY OF THE UNIFORMLY TWISTED FLUX TUBE

In order to compute the helicity of the magnetic field given by Eq. (5), the vector potential must be determined. By symmetry, $\mathbf{A}$ is not a function of ϕ or z and $A_r=0$. $\mathbf{B}=\nabla\times\mathbf{A}$ is readily integrated to give

$$A_\phi=\frac{B_0a^2}{2r}\ln(a^2+r^2)\ ,\qquad A_z=-\frac{B_0a}{2}\ln(a^2+r^2)\ . \tag{7}$$

The singular nature of these potentials is due to the fact that the current density [related by Eq. (4) to the field] is not constrained to a finite volume. Such singular potentials often occur in magnetostatics [17].

If one computes the integrand of Eq. (1), it is found that it vanishes since $\mathbf{A}\cdot\mathbf{B}=0$. In fact, the vanishing of the helicity does not require the explicit form of the potential given by Eqs. (7). The potential obtained by integrating $\mathbf{B}=\nabla\times\mathbf{A}$ could be left in integral form and subsequently used to show that $A_\phi/A_z=-a/r$ (which means, incidently, that the distance along the z axis traversed by $\mathbf{A}$ in one turn about the axis is independent of r and is the same as that traversed by $\mathbf{B}$). Since $B_\phi/B_z=r/a$, $\mathbf{A}\cdot\mathbf{B}=(0,-A_za/r,A_z)\cdot(0,B_zr/a,B_z)=0$. Yet the field lines are clearly linked and therefore the helicity cannot vanish.

The resolution to this apparent paradox lies in two additional factors:

(i) The vector potential computed in Eqs. (7) is determined only up to the gradient of a scalar function $\nabla\chi$ which leads to an additional helicity term $\int\nabla\chi\cdot\mathbf{B}dV$.

(ii) For force-free fields satisfying Eq. (4), the field lines lie on the surfaces $\alpha=$const (provided α is not globally constant). This can be seen by taking the divergence of Eq. (4) to obtain $\nabla\alpha\cdot\mathbf{B}=0$. Cowling's theorem [18] then tells us that these surfaces cannot be simply connected.

There is a theorem by Hopf [19] which states that the torus and the Klein bottle are the only smooth, compact, connected surfaces without boundary that can have a nonsingular (nowhere vanishing) vector field. Nonsingular magnetic surfaces can therefore be expected to have the topology of nested tori. If α is a constant, the field can have more complicated topologies [20].

The solution given by Eq. (5) to the force-free field equation can be given the topology of a torus by identifying $z=\pm z_0$. The means of evaluating the additional term $\int\nabla\chi\cdot\mathbf{B}dV$ when the volume of integration is a torus has been given elsewhere [21]. The helicity introduced by this term is

$$H=\int_{\mathrm{v}}\nabla\chi\cdot\mathbf{B}dV=\int_{\mathrm{v}}\nabla\cdot(\chi\mathbf{B})dV=\int_{\partial\mathrm{V}}(\chi\mathbf{B})\cdot d\mathbf{S}+\int_{\Sigma_1}[\chi]\mathbf{B}\cdot d\boldsymbol{\Sigma}_1\ , \tag{8}$$

where $[\chi]$ is the jump across Σ_1 (see Fig. 2). Since the bounding surface of the torus is a magnetic surface, the first term on the right-hand side vanishes. It can be shown that $[\chi]$ is constant on Σ_1. Thus, Γ can be taken as bounding Σ_2 and $[\chi]$ can be taken outside the integral.

Now for any two points p_1 and p_2, the difference in the value of the scalar function χ is $\chi_{p_2}-\chi_{p_1}=\int_{p_1}^{p_2}\nabla\chi\cdot d\mathbf{l}$. If the path of integration is closed, but does not cross a cut needed to make χ single valued, the integral will vanish by Stokes' theorem. When the path of integration Γ does cross the cut Σ_1, the value of the jump $[\chi]$ is [22]

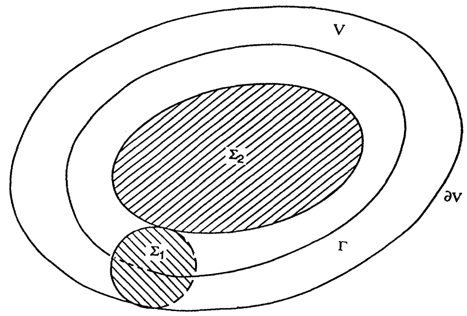

FIG. 2. The solution given by Eq. (5) can be given the topology of a torus by identifying $z=\pm z_0$. The domain V where the force-free condition holds is interior to the torus. Since $[\chi]$ is constant on Σ_1, Γ may be taken as bounding Σ_2.

$\int_\Gamma \nabla\chi\cdot d\mathbf{l}=\int_{\Sigma_2}\mathbf{B}\cdot d\mathbf{\Sigma}_2=\Phi_{\Sigma_2}$. Thus,

$$H=\int_{\Sigma_1}[\chi]\mathbf{B}\cdot d\mathbf{\Sigma}_1=\int_\Gamma \nabla\chi\cdot d\mathbf{l}\int_{\Sigma_1}\mathbf{B}\cdot d\mathbf{\Sigma}_1=\Phi_{\Sigma_1}\Phi_{\Sigma_2}\ . \quad (9)$$

This example is interesting because the helicity explicitly depends on the topology. Without the gauge term, and the multiply connected nature of the space, the helicity would vanish. It is also an explicit physical realization of the exchange of twist helicity for link helicity.

APPLICATION TO SOLAR PROMINENCE MODELS

Consider a flux tube emerging from the sun's photosphere into the chromosphere to form an arch. In the photosphere or below, material motions dominate and the magnetic-field structure will be assumed to be unknown; in the chromosphere, and above in the corona, the gas pressure is small and the field must be approximately force free (it is common to consider potential fields, where the current vanishes, to be force free with vanishing α; linear force-free fields to be those with α equal to a constant; and nonlinear force-free fields to be those having α a function of position). The magnetic field associated with a solar prominence is often modeled as a twisted flux tube. Twisting can be introduced either by photospheric twisting motions (due to Coriolis forces) at the locations where the base of the arch enters the photosphere [23], or by flux cancelation; i.e., by shear and convergence along the neutral line separating two regions of opposite magnetic polarity, followed by reconnection of the field lines [24]. In either case, currents are introduced that flow along the field lines in the force-free region.

Parker [16] has forcefully argued that one cannot expect to find in nature flux ropes with a structure more complex than a more or less uniform twist across the radius of the tube. Henceforth, for purposes of discussion, the force-free, uniformly twisted solution given by Eq. (5) will be taken as an appropriate model for the magnetic field associated with a solar prominence. The field in the flux tube is confined by a pressure differential across its boundary. Since the flux along the axis of the tube is conserved, the decrease in the mean longitudinal field as the tube is twisted leads to an increase in the radius of the tube. Physically, for an axially symmetric solution such as that given by Eq. (5) to be a reasonable approximation to the twisted flux tube used to model the field associated with a prominence, the aspect ratio (the ratio of the principal to minor radius) of the prominence must be large. Note also that the use of this solution to form a flux tube of finite radius R means that there will be an additional surface current needed to match to any potential field that may be present for $r>R$.

Helicity in the prominence

Since the field configuration below the photosphere is assumed to be unknown, it is clear that only relative helicities in the sense of Berger and Field, where the field below the photosphere is assumed to be the same for different field configurations in the force-free region above the photosphere, can be computed. For the uniformly twisted flux tube, $\mathbf{A}\cdot\mathbf{B}$ vanishes in the volume of integration above the photosphere (the flux tube). Consider now the relative helicity between a uniformly twisted flux tube and a potential field (consistent with the approximation made here, simply a field along the axis of the tube). By symmetry, $\mathbf{A}\cdot\mathbf{B}$ also vanishes for this field. It is clear from the previous discussion that the nonvanishing of the helicity depends on a gauge term and the fact that the space consisting of the interior of the flux tube and the region below the photosphere is multiply connected.

Since, in the region below the photosphere, the fields of the twisted flux tube and the untwisted one are assumed to be the same, the vector potential for each can differ at most by a gauge term $\nabla\chi$. This term can also be allowed to absorb any gauge freedom in the region above the photosphere. Thus the relative helicity reduces to the right-hand side of Eq. (9) and is consequently given by

$$\Delta H=\Phi_{\Sigma_1}\Phi_{\Sigma_2}\ . \quad (10)$$

Here Φ_{Σ_1} is the (conserved) flux within the tube and Φ_{Σ_2} is the flux through the surface bounded by the arch and the photosphere (Fig. 2 is applicable).

Magnetic energy contained in the prominence

Knowledge of the field configuration in the force-free region is insufficient to determine the field energy. This follows from the fact that $\mathbf{H}$ in the force-free relation, Eq. (4), is indeterminate up to an arbitrary multiplicative constant. It is, however, possible to compute relative energies. For example, one might be interested in the energy difference between a twisted arch and a similar arch described by a potential field. The solution of Eq. (5) again gives an interesting example.

Consider the energy of a uniform field B_0 in an untwisted cylindrical flux tube of radius R_0. As the tube is twisted the radius increases to R. Ignoring for the moment any contribution from surface terms, the magnetic energy in the tube is

$$E=\frac{1}{\mu_0}\int_V (B_z^2+B_\phi^2)dV=\frac{B_0^2\pi z}{\mu_0}\int_0^R \frac{2rdr}{(1+r^2/a^2)}$$
$$=\frac{B_0^2}{\mu_0}\pi z a^2 \ln(1+R^2/a^2)\ . \quad (11)$$

On the other hand, the axial flux in the tube is given by

$$\Phi=2\pi B_0\int_0^R \frac{rdr}{(1+r^2/a^2)}=B_0\pi a^2\ln(1+R^2/a^2)\ . \quad (12)$$

The energy per unit length of the tube is then $(1/\mu_0)B_0\Phi$. Since the axial flux is constant and equal to the original flux $B_0\pi R_0^2$, the energy contained in the volume of the flux tube is independent of the twist. This does not, however, mean that energy cannot be stored in uniformly twisted magnetic fields. Twisting the field introduces an axial current density given by $J=\alpha H$ since the field is force free. Uniformly twisting a flux tube then results in a current along the tube that creates a magnetic field that links the tube. Using the expressions given above for α and H_z, the current is

$$I=\frac{4\pi B_0}{a\mu_0}\int_0^R \frac{rdr}{(1+r^2/a^2)^2}$$
$$=\frac{2\pi B_0 a}{\mu_0}[1-(1+R^2/a^2)^{-1}]\ . \quad (13)$$

Interestingly enough, there is a value of a that maximizes the current. The conservation of flux implies that

$$(1+R^2/a^2)=e^{R_0^2/a^2}\ . \quad (14)$$

The current can then be written as

$$I=\frac{2\pi B_0 a}{\mu_0}(1-e^{-R_0^2/a^2})\ . \quad (15)$$

Differentiating to find the extremum of the current I with respect to a results in the trancendental equation

$$1-e^{-1/u^2}(1+2/u^2)=0\ , \quad (16)$$

where $u=a/R_0$. This can be solved numerically to yield $u=0.89$. Using this value in the second derivative of the current with respect to a confirms that the extremum is a maximum. The value obtained for u falls into the stable range given by Parker [16] as $u\geq 0.5956$.

In writing Eq. (11), the surface term was ignored. In general, the expression for the energy is given by

$$E=\tfrac{1}{2}\int_V \mathbf{H}\cdot(\nabla\times\mathbf{A})dV+\tfrac{1}{2}\int_{\partial V}(\mathbf{H}\times\mathbf{A})\cdot d\mathbf{S}\ . \quad (17)$$

As has been shown above, the energy contained in the volume of the flux tube is independent of twist. The first term on the right-hand side of this equation will therefore not contribute when computing the difference in energy between a flux tube containing a potential field and one having a uniform twist. For a simply connected domain, the integrals in Eq. (17) are gauge invariant. When the domain is multiply connected, the gauge term takes on a physical significance. The surface term does not vanish for the twisted tube, but represents the energy in the field exterior to the domain of integration due to currents within the domain [21]. It yields the value $\frac{1}{2}I\Phi_{\Sigma_2}$, where I is given by Eq. (13) and Φ_{Σ_2} is again the flux through Σ_2.

As was the case for helicity, introducing a uniform twist into a flux tube does not increase the magnetic energy within the flux tube itself, but creates a current which acts as the source for a field that links the tube. The ambiguity as to whether the energy resides in the current or the field is an old one in magnetostatics.

SUMMARY

While the helicity of a magnetic field can be interpreted as a measure of the degree of linkage of the field lines, it can also have a nonlocal character. The example of a uniformly twisted, force-free flux tube was used to show how this element of nonlocality represents a physical manifestation of the exchange of twist helicity for link helicity. In this example, the nonvanishing of the helicity is explicitly due to a gauge term and the multiple connectedness of the domain containing the field lines. Similarly, the energy difference resulting from the twist depends on a boundary term that represents the energy in the link field.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy under Contract No. W-31-109-ENG-38.

APPENDIX

There are several definitions of the Hopf invariant [25]. Perhaps the simplest, geometrically, is to consider a sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ given by the equation $x_1^2+x_2^2+x_3^2+x_4^2=1$. Identify a point on this sphere with two complex numbers (z_1,z_2), where $z_1=x_1+ix_2$ and $z_2=x_3+ix_4$. Then the equation of the sphere is $|z_1|^2+|z_2|^2=1$; i.e., $\mathbb{S}^3=\{(z_1,z_2)\,|\,|z_1|^2+|z_2|^2=1\}$.

The Hopf mapping π: $\mathbb{S}^3\rightarrow\mathbb{S}^2$ is given by $\pi(z_1,z_2)=z_1/z_2$ and

$$\pi(z_1,z_2)\rightarrow[2\,\mathrm{Re}(z_1\overline{z_2})\ ,2\,\mathrm{Im}(z_1\overline{z_2}),|z_1|^2-|z_2|^2]\in\mathbb{S}^2\ . \quad (A1)$$

Now if $\lambda=e^{i\theta}$, $0\leq\theta\leq 2\pi$, $\pi(\lambda z_1,\ \lambda z_2)=\pi(z_1,z_2)$. Therefore, $\pi^{-1}(u)\cong\mathbb{S}^1$ for any $u\in\mathbb{S}^2$. The Hopf mapping can be considered to be the projection of the Hopf bundle, i.e., of the locally trivial fiber space whose total space $\mathbb{S}^3$ has a base space $\mathbb{S}^2$ and fiber $\mathbb{S}^1$. Given any two points $u,v\in\mathbb{S}^2$, the "circumferences" $\pi^{-1}(u)$ and $\pi^{-1}(v)$ are disjoint and linked in $\mathbb{S}^3$. The classical Hopf invariant is the linking number $\mathbb{L}(\pi^{-1}(u),\pi^{-1}(v))$.

That $\pi^{-1}(u)$ and $\pi^{-1}(v)$ are disjoint and linked can be readily seen by choosing, for example, $u=(1,0,0)$ and $v=(-1,0,0,)$ in $\mathbb{S}^2$ and using Eq. (A1) to obtain the additional relations $2z_1\overline{z}_2=1$ for u and $2z_1\overline{z}_2=-1$ for v. These are satisfied, respectively, by

$$z_1=z_2=\frac{e^{i\theta}}{\sqrt{2}}\ \text{ and }\ z_1=-z_2=\frac{e^{i\theta}}{\sqrt{2}}\ , \quad (A2)$$

where $z_1, z_2 \in \mathbb{S}^3$. Visualizing the linkage is aided by using the stereographic projection Π: $\mathbb{S}^3 \rightarrow \mathbb{R}^3$, with the projection point, which will be chosen here to be $(0,0,0,-1) \in \mathbb{S}^3$, identified with the point at infinity. That is,

$$\zeta_i = \frac{x_i}{1+x_4} , \tag{A3}$$

where $\zeta_i \in \mathbb{R}^3$, $x_i \in \mathbb{S}^3$ and $x_4 \neq -1$. This results in

$$\Pi[\pi^{-1}(u)]: \; \zeta_1 = \zeta_3 = \frac{\cos\theta}{\sqrt{2}+\sin\theta} \; ; \; \zeta_2 = \frac{\sin\theta}{\sqrt{2}+\sin\theta} ,$$
$$\Pi[\pi^{-1}(v)]: \; \zeta_1 = -\zeta_3 = \frac{\cos\theta}{\sqrt{2}-\sin\theta} \; ; \; \zeta_2 = \frac{\sin\theta}{\sqrt{2}-\sin\theta} . \tag{A4}$$

If these curves are plotted as a function of θ in the mutually orthogonal planes $\zeta_1 = \pm\zeta_3$, one obtains two disjoint and linked ellipses [26].

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